Another Topological View of Curves and Surfaces Inspired by 2D and 3D Locus Problems

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Abstract

This is an expanded paper from [14]. The concepts are motivated by some locus problems discussed in [9], and further in [10] and [11] for 2D and 3D cases, we explore whether two topological manifolds are topologically equivalent with technological tools in various scenarios. We derive some interesting facts about topological equivalence that may be used in the classroom.

1 Introduction

In a series of papers, [12, 9, 10, 11], we explored problems based on characterizing the loci, in 2D and 3D, generated by moving points that satisfy certain constraints. These problems were derivatives of an original problem that in turn came from some practising exercises for college entrance examination in China.

In particular, in [10], we discussed the following 3D locus problem:
Suppose we are given a point $A = (a_1, a_2, a_3) \in \mathbb{R}^3$ *and a generic point* $C = (c_1, c_2, c_3)$ *on a surface* $\Sigma \subset \mathbb{R}^3$. *Let* l *be the line in* $\mathbb{R}^3$ *passing through* A *and* C. *Suppose the line* l *intersects a well-defined point* $D = (d_1, d_2, d_3)$ *on the surface* Σ. *Thus, we want to determine the locus surface generated by a point* $E = (e_1, e_2, e_3) \in \mathbb{R}^3$, *lying on the line segment* CD, *and satisfying the condition* $\overrightarrow{ED} = s\,\overrightarrow{CD}$, *for a parameter* $s \in \mathbb{R}$, *when*

1. *the point A moves in a certain permissible manner;*
2. *the points C, or D on Σ are moved in a definite manner; and*
3. *the parameter s changes continuously in some interval of real numbers.*

Note that the action of moving the point A is equivalent to that of $D \in \Sigma$ since this last point is determined by A and C, through line l. Thus, in some situations we can define the point D as the "antipodal" point of C, relative to the point A. We will illustrate this situation with several examples.

Notice that if we denote by O the origin in $\mathbb{R}^3$, then, from the parallelogram law, we conclude that this problem is equivalent to that of determining the locus generated by the point E satisfying

$$\overrightarrow{OE} = s\,\overrightarrow{OC} + (1-s)\,\overrightarrow{OD}. \tag{1}$$

See Figure 1 for a better visual understanding of the problem.

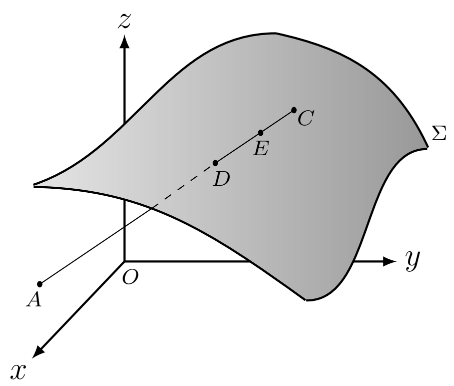


Figure 1: A 3D locus problem

In this paper, we continue exploring some scenarios related to this 3D locus problem, as described, but with an additional concern, namely, we are interested in determining the topological equivalence for different geometric loci generated by varying the parameter s in equation (1). Needless to say, the use of technological tools is a paramount part of our explorations and this gives rise to some interesting situations, worthy of investigation.

Thus, the plan for the paper is the following: In Section 2 we review some of the possible scenarios related to the settings that can be derived from equation (1) for different values of parameter s, and present some interesting examples. In Section 3, we introduce some topological concepts in order to discuss the topological equivalence of some of the generated loci, both in 2D and 3D. Additionally, we furnish with some examples. In Section 4 we continue the discussion of topological equivalence, but for some concrete examples.

It is important to remark that the use of technological tools is a relevant part in all the explorations and investigations presented in the paper. Thus, we provide all the computer programs and activities in the section of Supplementary Electronic Materials. The software used here corresponds to Computer Algebra Systems, such as Maple™ ([2]), and Maxima ([3]), alongside with Dynamic Geometry Software such as GeoGebra ([1]). All of the electronic materials provided are intended to give the reader a deeper insight and understanding of the different scenarios addressed in the paper.

2 Some curves and surfaces derived from locus problems

Equation (1) is a good start to explore some scenarios that can occur, and are directly related to the aforementioned locus problem. However, we must first define some terminology, specifically that of antipodal point, and the types of surfaces we shall be dealing with.

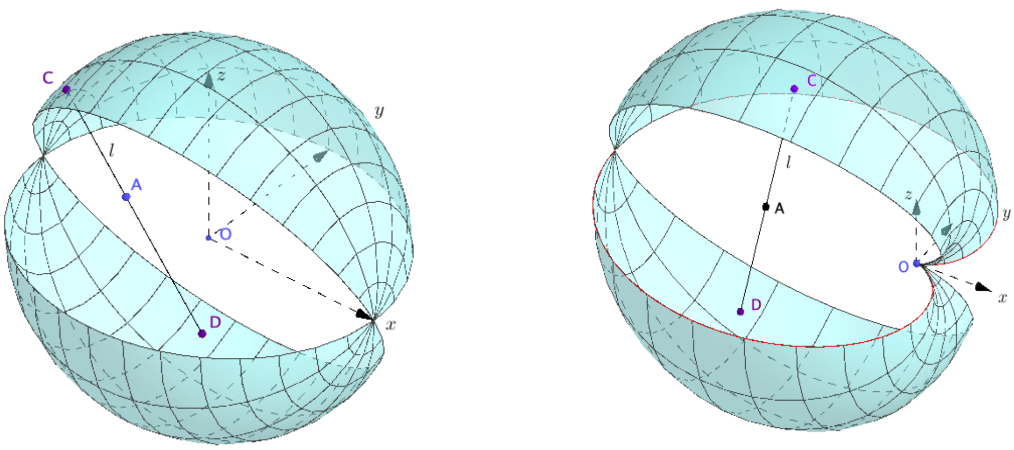
2.1 Some definitions

Definition 1 *Let* $R = \{(u, v) \mid u, v \in \mathbb{R}\} \subset \mathbb{R}^2$ *be a domain. The* parametric surface Σ *defined by the injective, real-valued functions* $x = x(u, v)$, $y = y(u, v)$, *and* $z = z(u, v)$ *is the set*

$$\Sigma = \{(x(u, v), y(u, v), z(u, v)) \mid (u, v) \in R\} \subset \mathbb{R}^3.$$

Usually, the coordinate functions in Definition 1 are assumed to be sufficiently smooth (in the sense of differentiability), and so the parametrized surface is a smooth one. Moreover, most of the surfaces in the examples in this paper are *closed surfaces*, that is, they are compact and without boundary; examples of which are the sphere, the torus, the Klein bottle, etc. In addition, if the tangent space of every point lies on the same side of Σ, we call Σ a convex surface. Unless otherwise specified in this paper, we assume that the surface $\Sigma \subset \mathbb{R}^3$ is closed and convex.

Definition 2 *Let* $\Sigma \subset \mathbb{R}^3$ *be a closed and convex surface such that* A *is a point in the interior of* Σ. *Let* C *be an arbitrary point on the surface* Σ *and let us denote by* l *the straight line through points* A *and* C. *We say that a point* D *and* $D \neq C$ *in* Σ *is the* antipodal point *of* C *relative to* A *if* D *is the intersection of* l *and* Σ.



(a) Antipodal points on a sphere.

(b) Antipodal points on a surface of revolution.

Figure 2: Antipodal points

Figure 2(a) illustrates the situation described in Definition 2. Notice that when the surface Σ is symmetrical about the origin and $A = (0, 0, 0)$, then antipodal points are easy to calculate. Moreover, even in the case when the surface is not a convex one, so long as it is non-self intersecting the antipodal points can still be calculated because Definition 2 is taken relative to a given point inside the surface as shown in Figure 2(b).

Theorem 3 *Let $\Sigma \subset \mathbb{R}^3$ be a parametrized, closed and convex surface and let $A \in \mathbb{R}^3$ be an interior point to Σ, i.e., A is enclosed inside the surface. If $C \in \Sigma$ is an arbitrary point, then there exists a point $D \in \Sigma$ such that D is the antipodal point of C relative to A.*

Proof. Let $\varphi : R \to \mathbb{R}^3$ be a parametrization of surface Σ with $\varphi(R) = \Sigma$ and $\varphi(u, v) = (x(u, v), y(u, v), z(u, v)) \in \Sigma$, for all $(u, v) \in R$ in the sense of Definition 2. Since $C = (c_1, c_2, c_3) \in \Sigma$, there exists a point $(u_0, v_0) \in R$ such that $\varphi(u_0, v_0) = C$, and thus

$$x(u_0, v_0) = c_1, \qquad y(u_0, v_0) = c_2, \qquad z(u_0, v_0) = c_3.$$

On the other hand, the straight line connecting points C and A is given, in parametric form, by $l : C + t(C - A)$, for $t \in \mathbb{R}$. Thus, if $A = (a_1, a_2, a_3)$, then we need to find a value of the parameter t, say $t_0 \neq 0$, such that

$$\big((1 + t_0)c_1 - t_0a_1, (1 + t_0)c_2 - t_0a_2, (1 + t_0)c_3 - t_0a_3\big) \in \Sigma.$$

Now, consider the plane Π_{AOC} passing by the three points C, A, and the origin $O = (0, 0, 0)$. We can calculate the equation for this plane in the form

$$ax + by + cz + d = 0.$$

The plane Π_{AOC} contains line l and defines a closed curve on the surface Σ, say $\alpha = \alpha(s)$, which is the intersection of Σ with Π_{AOC}. Here $s \in \mathbb{R}$ is the parameter that defines the curve $\alpha : I \to \mathbb{R}^3$ for some interval $I \subset \mathbb{R}$. Therefore, there exist values of the parameter s, say s_1 and s_2 such that $\alpha(s_1) = C$ and $\alpha(s_2)$ is a second point of intersection of line l, plane Π_{AOC} and the surface Σ. We denote this point by D: $\alpha(s_2) = D = (d_1, d_2, d_3) \in \Sigma$. We also have

$$ad_1 + bd_2 + cd_3 + d = 0,$$

and there exists $(u_1, v_1) \in D$ such that $\varphi(u_1, v_1) = (d_1, d_2, d_3)$ and we can solve for t_0 in any of the three equations

$$\begin{aligned}(1 + t_0)c_1 - t_0a_1 &= d_1,\\ (1 + t_0)c_2 - t_0a_2 &= d_2,\\ (1 + t_0)c_3 - t_0a_3 &= d_3.\end{aligned}$$

■

We illustrate some examples of this demonstration with a DGS (see [S1], [S2], and [S3]) in the Supplementary Electronic Materials.

2.2 Several scenarios for study

Following the above cited papers [7, 8, 9, 10, 11, 12], the locus constructions for surfaces are natural generalizations of those corresponding to curves in the plane. Thus, we also will be making use of several $2D$ examples in order to illustrate some of the concepts we will be dealing with. In particular, we will talk about antipodal points on curves and loci in $2D$.

Now, let us return to the situation of equation (1) in order to depict some scenarios that are worthy to explore, which are the main motivations for this paper, and these will be addressed in what follows.

1. Suppose we fix both the point $C \in \Sigma$ and the parameter s in (1). Let D be the antipodal point of C, relative to the fixed point A, when connected by a line l through points A and C. Thus, when we consider the locus of E satisfying $\overrightarrow{ED} = s\,\overrightarrow{CD}$, which is equivalent to (1), it is not difficult to figure out that the locus E represents a shifted and scaled surface from Σ. In fact, think of D as a point freely moving on the surface and which is joined with E through a line segment.

2. Clearly, from the above description of the problem, there is a direct dependence of the locus generated by E in (1) on the points C and D, even as the parameter s varies. We shall see how a curve or surface determined by the point C is transformed to another curve or surface determined by D.

In what follows, we shall explore each one of the situations described above. Nevertheless, as a motivation of what we intend to do, we provide an illustrative $2D$ example in the following subsection.

2.3 Scaled and shifted locus

Suppose $\Sigma \subset \mathbb{R}^3$ is a regular surface, and let C be a fixed point on Σ. For any point $A \neq C$, let D be the antipodal point of C on the surface Σ when connected by a line l through points A and C. We notice that locus of the point E satisfying $\overrightarrow{OE} = s\,\overrightarrow{OC} + (1-s)\,\overrightarrow{OD}$ can be characterized as:

1. A dilation carried out by the point D for $1 - s > 0$. When this number is negative, the locus is a reflection, $|1-s|D$, of D about the origin.

2. It is a translation defined by the vector $s\,\overrightarrow{OC}$.

We remark that a transformation of a curve c to any other curve is not unique. This then gives rise to an opportunity for us to talk about topological transformations of curves and surfaces – a topic that we shall address in the upcoming section.

3 Locus and topological transformations

Recall that a topological space is a non-empty set together with a topological structure that is given by the open sets of the topology. A non-empty set X can be equipped with different topological structures, and any subset $A \subset X$ can be endowed with an inherited topological structure, namely, the *relative topology*: the open sets in the relative topology on A are exactly the subsets of the form $A \cap O$ for some open sets O of X.

For the spaces $\mathbb{R}^n$, with coordinates $(x_1, \ldots, x_n)$, the usual topological structure is given by a base of open balls, which is defined by the metric space structure of $\mathbb{R}^n$. Recall that the *metric function* $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}$, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ makes $(\mathbb{R}^n, d)$ a *metric space*. This metric gives rise to open balls $B_\varepsilon(x) = \{y \in \mathbb{R}^n \mid d(x, y) < \varepsilon\}$, where ε is a positive real number, which are the basic open sets for the usual topology on $\mathbb{R}^n$.

Thus, any curve or any hyper-surface in $\mathbb{R}^n$ can be viewed as topological spaces with their respective relative topologies, acquired from their environment space. For the cases we are interested in, the environment spaces will be simply $\mathbb{R}^2$ or $\mathbb{R}^3$; thus, we can view these curves and surfaces as relative topological spaces on which we can apply topological tools that are introduced in what follows.

Now, let us recall a fundamental concept in topology, namely that of *continuity*. Let (X, τ_X) and (Y, τ_Y) be two topological spaces, with τ_X and τ_Y their respective topologies. A function $f : X \longrightarrow Y$ is *continuous* if for any open set $O \subset Y$ it holds that $f^{-1}(O) \subset X$ is open in X. Here, the notation $f^{-1}(U)$ denotes the inverse image, in X, of $U \subset Y$.

When f is bijection, then it has an inverse $f^{-1} : Y \longrightarrow X$ defined by: for any $y \in Y$, $f^{-1}(y) = x \in X$ if and only if $f(x) = y$. We say that f is a *homeomorphism* if it is a continuous bijection and with f^{-1} also being continuous. In this situation, X and Y are said to be *homeomorphic* topological spaces.

Homeomorphisms between topological spaces are also called *continuous transformations*, or *topological transformations*. Intuitively, these transformations can be thought of as functions that transforms points of one space, that are arbitrarily close to each other, onto points of another space that are also arbitrarily close. Spaces that are related in this way are said to be *topologically equivalent*. In a technical sense, we have to be careful with the notion of closeness since it only makes sense when we can measure distances in a space, so we need to have a metric structure. Nevertheless, that is what we meant by "intuitively".

If a space or some region in a space is transformed into another equivalent space or region by bending, stretching, etc., the change is a special type of topological transformation called a *continuous deformation*. Two figures (e.g., certain types of knots) may be topologically equivalent, however, without being changeable into one another by a continuous deformation.

It is intuitively evident that all simple closed curves in the plane and all closed polygons are topologically equivalent to a circle. Similarly, all closed cylinders, closed cones, convex polyhedra, and other simple closed surfaces are equivalent to a sphere. On the other hand, a closed surface such as a torus (a doughnut) is not equivalent to a sphere, since no amount of bending or stretching can get rid of the hole in the torus to make it into a sphere. Similarly, a surface with a boundary is not topologically equivalent to a sphere, e.g., a cylinder with an open top, which may be stretched into a disk (a circle plus its interior).

In view of the locus generated in the settings of our problem by a point E satisfying (1), see page 2: for a given fixed A, we intend to find the maximum value of s such that the locus is a simple closed curve in $2D$ or a surface that is topologically equivalent to a sphere in $3D$. Subsequently, we extend the idea of finding a new surface generated by two or more surfaces.

3.1 Transforming topologically one simple closed curve into another

In the settings of our explorations, we will show how a simple closed curve such as an ellipse or a surface like a sphere can be transformed to another simple closed curve or into another surface that is topologically equivalent to the sphere. First, we consider the following elementary $2D$ scenarios.

Example 4 *Let us denote by* $c = c(u) = \big[x(u), y(u))\big] = \big[(1-\cos u)\cos u, (1-\cos u)\sin u\big] \in \mathbb{R}^2$ *the parametrized cardioid curve, with* $u \in [0, 2\pi]$*. Next, denote by* $d = d(u) = [\widetilde{x}(u), \widetilde{y}(u)] = \big[a\cos u, b\sin u\big] \in \mathbb{R}^2$ *the curve that parametrizes an ellipse with semi-axes* a *and* b*. In Figure 3 both parametrized curves are illustrated and the ellipse's semi-axes are set as* $a = 2$ *and* $b = 3/2$*.*

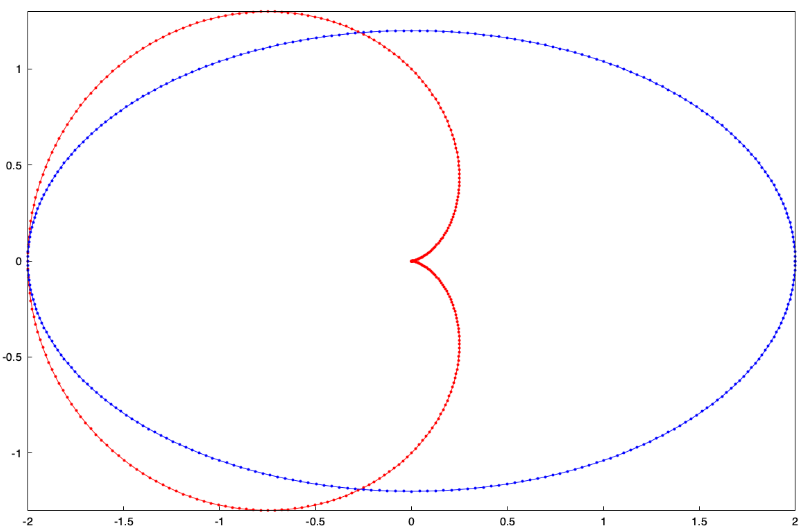


Figure 3: A cardiod together with an ellipse with semi-axes $a = 2$, $b = 1.5$

Now let us pick any arbitrary point C on the curve c, and similarly, let us select any point D on the ellipse d. For a parameter $s \in [0, 1]$, define the transformation T

$$\begin{pmatrix} s \\ u \end{pmatrix} \xrightarrow{T} \begin{bmatrix} F(s,u) \\ G(s,u) \end{bmatrix}$$

where

$$\begin{bmatrix} F(s,u) \\ G(s,u) \end{bmatrix} = s \begin{bmatrix} x(u) \\ y(u) \end{bmatrix} + (1-s) \begin{bmatrix} x'(u) \\ y'(u) \end{bmatrix}. \tag{2}$$

Notice that if $O = (0, 0)$ denotes the origin in $\mathbb{R}^2$, and setting

$$OE = \begin{bmatrix} F(s,u) \\ G(s,u) \end{bmatrix}, \qquad OC = \begin{bmatrix} x(u) \\ y(u) \end{bmatrix}, \quad \textit{and} \quad OD = \begin{bmatrix} x'(u) \\ y'(u) \end{bmatrix}$$

then (2) is nothing but the well-known relationship (1):

$$OE = s\,OC + (1-s)\,OD,$$

which allow us to express (2) in explicit terms as

$$OE = \begin{bmatrix} F(s,u) \\ G(s,u) \end{bmatrix} = s\,OC + (1-s)\,OD = s \begin{bmatrix} (1-\cos u)\cos u \\ (1-\cos u)\sin u \end{bmatrix} + (1-s) \begin{bmatrix} a\cos u \\ b\sin u \end{bmatrix}. \tag{3}$$

Clearly, for $s = 1$ we have that the image of transformation T is the cardioid c and for $s = 0$ the image of T is the unit ellipse d. Also, it is clear that T is a continuous transformation that transforms, point by point, the given ellipse into the cardioid as s varies from 0 to 1. Accordingly, T^{-1} is a continuous transformation from the cardioid to the ellipse. Therefore, a simple closed cardioid is topologically equivalent to an ellipse.

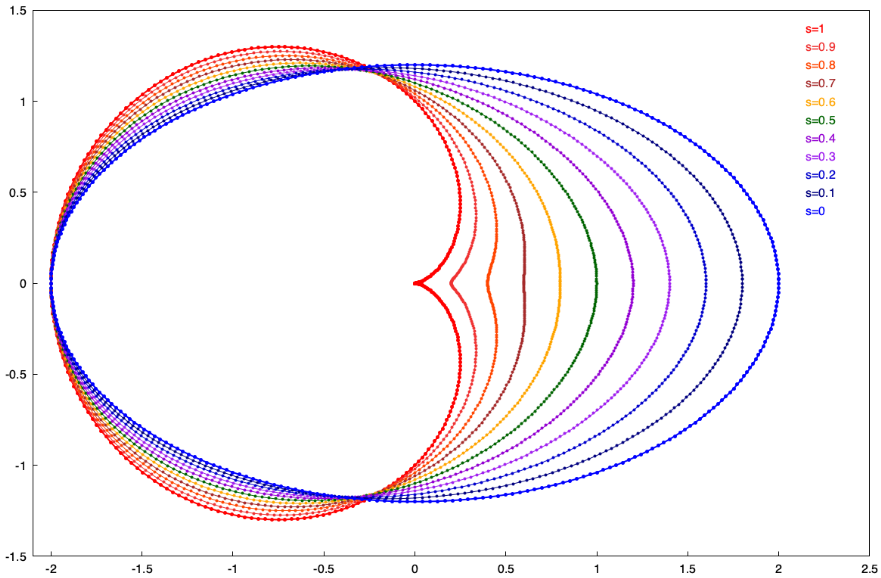


Figure 4: A continuous deformation of a cardiod into an ellipse

In Figure 4, we can appreciate how the cardioid is transformed, according to (2) and (3), into an ellipse, for different values of the parameter s. Note that the symmetry about the x-axis is preserved [See [S4] for exploration].

In Figure 5 we present an animation where we clearly see the process of transforming topologically a cardiod into an ellipse. In order to render this animation smoothly, we recommend to open the corresponding PDF output file with Adobe Acrobat Reader™.

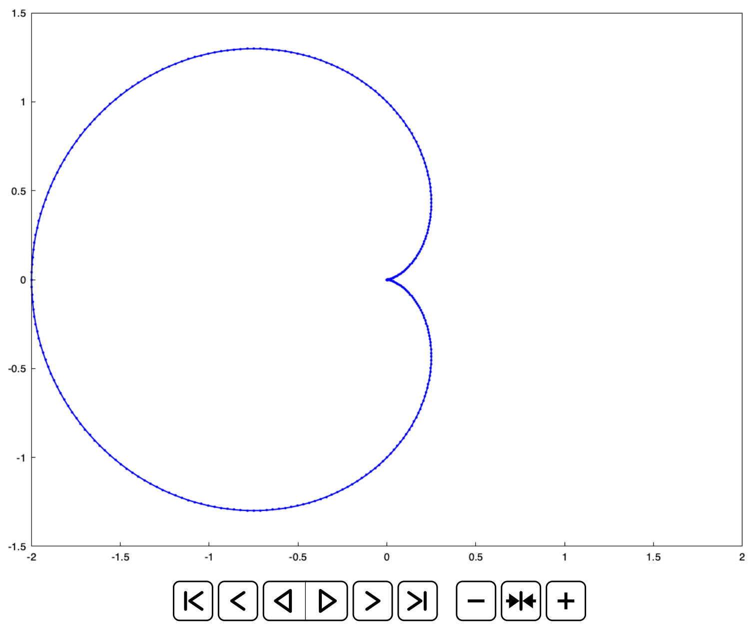


Figure 5: A cardiod is continuously deformed into an ellipse

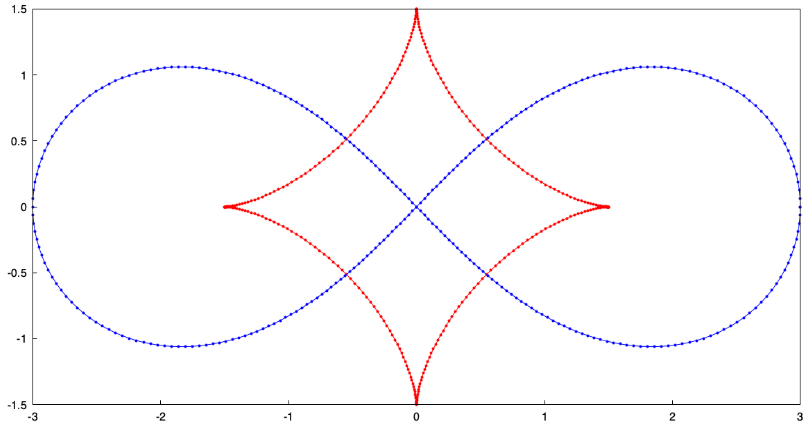


Figure 6: An astroid with $a = 3/2$, and a lemniscate with half-width $a = 3$.

Exercise: We leave as an exercise to the reader of finding an appropriate transformation between a circle and a cardioid that has being shifted to the right by one unit from the original one, given by the parametric functions $x(u) = (1 - \cos u)\cos u$ and $y(u) = (1 - \cos u)\sin u$. Readers would discover that the correct circle to be chosen for this transformation should be the one obtained by shifting the original unit circle to the right by one unit.

Example 5 *In this example we address the case of two quite famous plane curves, namely the astroid and the lemniscate. The astroid is a 4-cusped hypocicloid which is obtained by rolling a circle of radius $a/4$ on the inside a circle of radius a. It is well known that a parametrized astroid is given by*

$$c(u) = \big(a\cos^3 u, a\sin^3 u\big), \tag{4}$$

with $0 \le u \le 2\pi$.

On the other hand, a lemniscate is a closed curve that was studied in connection with the problem of isochrones by Johan Bernoulli in 1694. However, the lemniscate is a particular instance within a family of plane curves called Cassini ovals that were introduced by Cassini in 1680. The parametric equations for a lemniscate with half-width a is

$$d(u) = \left(\frac{a\cos u}{1+\sin^2 u}, \frac{a\cos u \sin u}{1+\sin^2 u}\right), \tag{5}$$

with $0 \le u \le 2\pi$. In Figure 6, we show both curves.

Again, our main tool is equation (1): $OE = s\,OC + (1-s)OD$, with C, and D, arbitrary points on curves c and d, respectively. Therefore, the locus generated by point E satisfies the following relation:

$$OE = \begin{bmatrix} F(u,s) \\ G(u,s) \end{bmatrix} = s\,OC + (1-s)OD = s\begin{bmatrix} a\cos^3 u \\ a\sin^3 u \end{bmatrix} + (1-s)\begin{bmatrix} \frac{a\cos u}{1+\sin^2 u} \\ \frac{a\cos u \sin u}{1+\sin^2 u} \end{bmatrix}. \tag{6}$$

It is clear, from (6), that for $s = 1$ we have the astroid, and for $s = 0$ we get the lemniscate. Thus, equation (6) is an example of an application that transforms an astroid into a lemniscate and it is, in addition, a homeomorphism. Therefore, this shows that the astroid is topologically equivalent to the lemniscate. See Figure 7 below for the animations generated using the parameter s. Also, we refer to [S5] for animation and [S6] of using the CAS [2] for exploration.

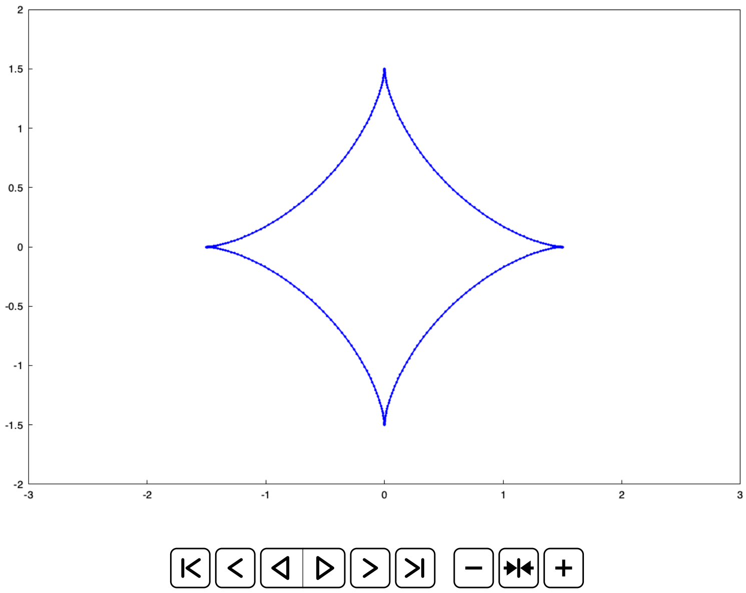


Figure 7: An astroid is transformed into a lemniscate.

Remark. Note that the continuous deformations defined in Examples 4 and 5 are of the same type, which is commonly known as linear homotopy.

3.2 Maximum value of parameter s for topological equivalence

As it is discussed in [10] and [11], the locus generated by the point E on an ellipsoid depends on the location of the fixed point A and the parameter s. In [11], we have proved that when A is the point at infinity and $s > 0$ is a finite real number, the locus is an ellipsoid, which we denote by $E(s)$, and is topologically equivalent to the sphere. However, when A is the point at infinity and $s \to \infty$, then the locus becomes an elliptical cylinder, say we denote it by $E_c(s)$. Therefore, there exists an s^* such that $E(s)$ is an ellipsoid when $s < s^*$, and is not topologically equivalent to an elliptical cylinder $E_c(s)$ when $s > s^*$. However, finding the critical value s^* requires further investigation.

In the remaining paper, unless otherwise is specified, we address the situation when the given fixed point A is **distinct from the point at infinity**.

4 Surfaces parametrized by longitude and latitude

Recall that any surface is a topological space whose topological structure is inherited from the ambient space $\mathbb{R}^3$ it lives in. Also, two surfaces are topologically equivalent if there exists a homeomorphism f between them, where these surfaces are viewed as two topological spaces. Such a homeomorphism f is a bijection with the property that both f and its inverse f^{-1} preserve open sets. If a certain geometric object is transformed into another topologically equivalent one by bending, stretching, etc., this process entails a special type of topological transformation called *continuous deformation.*

In fact, we could refer to the surfaces we are working with in this paper as *topological manifolds* since, as subsets of $\mathbb{R}^3$, they comply with all the conditions for that kind of manifolds: being Hausdorff topological spaces, with a covering of open sets, each one homeomorphic to an open set in $\mathbb{R}^3$ and in the intersections of the sets of the covering, the *transition functions* are also homeomorphisms from a subset of the surface onto another subset of the surface. Thus, in what follows we shall be dealing with surfaces that are topological manifolds and we will refer to them with such a terminology.

4.1 Surfaces of perturbing the longitude and latitude

Suppose S is a given parametrized surface, say

$$S = \{(f(u,v), g(u,v), h(u,v)) \mid u, v \in R\}, \tag{7}$$

for a rectangular domain $R \subset \mathbb{R}^2$, and real-valued continuous functions f, g, and h defined on R. Usually the values of the parameters will be $0 \leq u \leq 2\pi$, and $0 \leq v \leq \pi$, hence we will refer to them as *longitude* and *latitude*, respectively.

Now, with the same parametrization (7) for S we can consider some other surfaces, namely

$$S_1 = \{(f(u+\alpha_1, v), g(u+\alpha_2, v), h(u+\alpha_3, v)) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\}, \tag{8}$$

or

$$S_2 = \{(f(u, v+\beta_1), g(u, v+\beta_2), h(u, v+\beta_3)) \mid \beta_1, \beta_2, \beta_3 \in \mathbb{R}\}, \tag{9}$$

for real parameters α_i and β_i, for $i = 1, 2, 3$.

Note how the new parameters α_i and β_i change the longitude u and latitude v. This *perturbation* of the parameters turns out to be quite noticeable, even for *small* values of the colatitude parameters β_i, when using spherical or ellipsoidal coordinates for parametrizing the surface S, as we will see in what follows.

Recall that Cartesian coordinates (x, y, z) are related with spherical coordinates (ρ, θ, ϕ) by the rule

$$\begin{bmatrix} \rho \\ \theta \\ \phi \end{bmatrix} \longrightarrow \begin{bmatrix} x(\rho,\theta,\phi) \\ y(\rho,\theta,\phi) \\ z(\rho,\theta,\phi) \end{bmatrix} = \begin{bmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{bmatrix} \tag{10}$$

Here, and only for more clarity, points in $\mathbb{R}^3$ are expressed in column vector form. We will continue using this notation when it becomes necessary in what follows.

The Jacobian matrix corresponding to the change of coordinates (10) is given by

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{bmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & -\rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix},$$

whose determinant is

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = -\rho^2\sin\phi.$$

Hence, the Inverse Function Theorem can be applied everywhere except at $\rho = 0$, or $\sin\phi = 0$. In the case of a sphere with radius ρ, this occurs for the points $(0,0,0)$, $(\rho,0,0)$ and $(\rho,0,\pi)$. Thus, any neighborhood of a point (ρ_0,θ_0,ϕ_0), that does not contain any of the points $(0,0,0)$, $(\rho,0,0)$, nor $(\rho,0,\pi)$, is mapped homeomorphically onto a neighborhood of $\big(x(\rho_0,\theta_0,\phi_0), y(\rho_0,\theta_0,\phi_0), z(\rho_0,\theta_0,\phi_0)\big)$ by (10) (in fact, it is mapped diffeomorphically).

Therefore, in order for the transformation of variables given by (10) to be a bijection, we are constrained to the region in $\mathbb{R}^3$ defined by

$$\rho > 0, \qquad 0 \le \theta < 2\pi, \qquad 0 < \phi < \pi.$$

The example below demonstrates the importance to maintain values of the colatitude within the proper ranges.

Example 6 *Consider an sphere parametrized by spherical coordinates*

$$S = \big\{\big(f(u,v), g(u,v), h(u,v)\big)\big\} = \left\{ \begin{bmatrix} r\cos(u)\sin(v) \\ r\sin(u)\sin(v) \\ r\cos(v) \end{bmatrix} \mid u \in [0,2\pi), v \in (0,\pi), r > 0 \right\}, \quad (11)$$

with r the radius of the sphere, but here, for simplicity, we set $r = 1$.

Now, according to (8) and (9) we set the values of the parameters α_i's and β_i's as

$$\begin{aligned} \alpha_1 &= \pi/8, & \beta_1 &= \pi/12, \\ \alpha_2 &= \pi/10, & \beta_2 &= \pi/16, \\ \alpha_3 &= \pi/12, & \beta_3 &= \pi/20. \end{aligned}$$

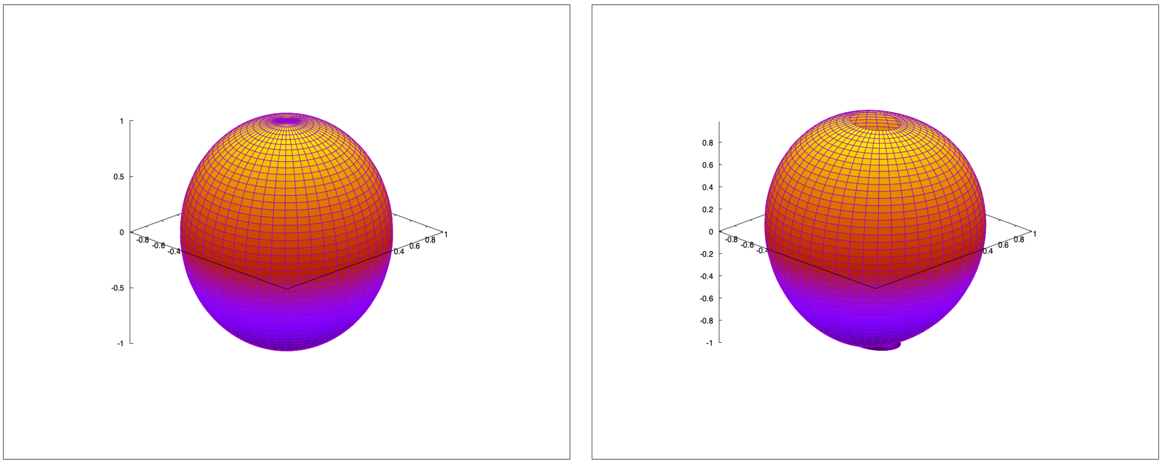
Therefore, we have

$$S_1 = \left\{ \begin{bmatrix} \cos(u+\pi/8)\sin(v) \\ \sin(u+\pi/10)\sin(v) \\ \cos(v) \end{bmatrix} \mid u \in [0,2\pi), v \in (0,\pi) \right\}, \quad (12)$$

and

$$S_2 = \left\{ \begin{bmatrix} \cos(u)\sin(v+\pi/12) \\ \sin(u)\sin(v+\pi/16) \\ \cos(v+\pi/20) \end{bmatrix} \mid u \in [0,2\pi), v \in (0,\pi) \right\}, \quad (13)$$

whose plots are shown in Figure 8. Notice that the surface S_2 in Figure 8b presents a hole at the north pole and a slight "bump" that appears at the south pole. Thus, those tearings which are clearly visible on this surface, destroy completely the topological properties of the original sphere (11).



(a) Perturbing a sphere on its longitude, as in (12).

(b) Perturbing a sphere on its latitude, as in (13).

Figure 8: Deformations of a sphere according to (12) and (13)

Moreover, if the original parametrized surface $S \subset \mathbb{R}^3$ in (7) is transformed into another surface according to

$$\begin{bmatrix} F(u,v,\alpha_1,\beta_1,s) \\ G(u,v,\alpha_2,\beta_2,s) \\ H(u,v,\alpha_3,\beta_3,s) \end{bmatrix} = s \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} + (1-s) \begin{bmatrix} f(u+\alpha_1, v+\beta_1) \\ g(u+\alpha_2, v+\beta_2) \\ h(u+\alpha_3, v+\beta_3) \end{bmatrix}, \tag{14}$$

then we could be interested in the topological properties of the new surface defined by the locus

$$\widetilde{S} = \left\{ \left(F(u,v,\alpha_1,\beta_1,s), G(u,v,\alpha_2,\beta_2,s), H(u,v,\alpha_3,\beta_3,s) \right) \right\}. \tag{15}$$

Notice that for $s = 0$, in (14), we get the surface

$$\widetilde{S} = \left\{ \left(f(u+\alpha_1, v+\beta_1), g(u+\alpha_2, v+\beta_2), h(u+\alpha_3, v+\beta_3) \right) \right\}, \tag{16}$$

in (15), while for $s = 1$ we obtain the original surface S in (7).

For the unit sphere, transformation (14) produces a new surface, namely

$$\widetilde{S} = \left\{ s \begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) \end{bmatrix} + (1-s) \begin{bmatrix} \cos(u+\pi/8)\sin(v+\pi/12) \\ \sin(u+\pi/10)\sin(v+\pi/16) \\ \cos(v+\pi/20) \end{bmatrix} \right\}, \tag{17}$$

and the results are quite remarkable and are a direct consequence of perturbing both, the longitude and the latitude. These results for the unit sphere are shown in the animation presented in Figure 9. Notice how the sphere has undergone several changes that completely alter its topological nature.

Detailed descriptions of all these explorations can be found in the Supplementary Materials, and they might prove to be useful when one conducts further investigations along this vein. On the other hand, although we have not yet demonstrated what it is behind the singularities occurring in surface S_2, in (9), as a consequence of the perturbations in the latitude, we conjecture that those behaviors are related to the fact that the values of the latitude parameters are far beyond the proper ranges when using spherical coordinates for parametrizing the surface.

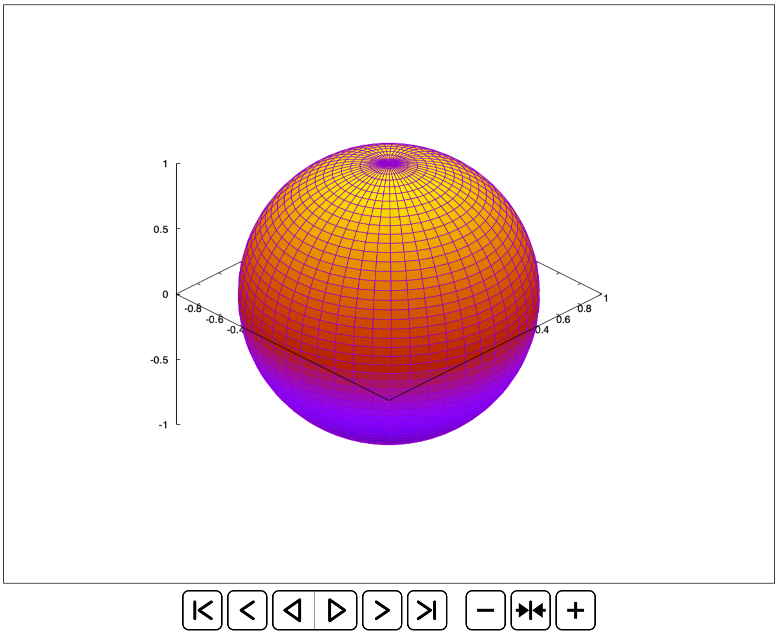


Figure 9: Transformation of a sphere according to (17).

We conclude this subsection by saying that when a surface S is transformed into another one by the mechanisms of perturbing the latitude and colatitude, given by (8) and (9), the resulting surface may turn out not to be topologically equivalent to the former one. Here, the systems of coordinates play an important role as we will demonstrate in the next subsection.

4.2 Topologically equivalent and nonequivalent surfaces

In view of the examples presented in Subsection 4.1, we now explore some other cases, where a continuous transformation is applied to a surface to yield another, and ask if these two surfaces are topologically equivalent. Here we will present two more examples that may shed some light to our understanding.

Now, recall how our initial equation (1) was also applied in (3) and (6) to transform topologically one simple closed curve into another simple closed curve. In this part, we will apply (1) for exploring how a surface is transformed into another, resembling the processes described above in Subsection 3.1, for curves. Actually, taking s as an scaling factor, we can ask for the properties of a new surface defined as the locus $\big\{\big(F(u,v,\alpha_1,\beta_1,s), G(u,v,\alpha_2,\beta_2,s), H(u,v,\alpha_3,\beta_3,s)\big)\big\}$ generated by the points in $\mathbb{R}^3$ defined by:

$$\begin{bmatrix} F(u,v,\alpha_1,\beta_1,s) \\ G(u,v,\alpha_2,\beta_2,s) \\ H(u,v,\alpha_3,\beta_3,s) \end{bmatrix} = s \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} + (1-s) \begin{bmatrix} f(u+\alpha_1, v+\beta_1) \\ g(u+\alpha_2, v+\beta_2) \\ h(u+\alpha_3, v+\beta_3) \end{bmatrix}. \tag{18}$$

Hence, if we start with a surface $S \subset \mathbb{R}^3$ which is transformed, according to the transformation rule (18), into the surface $\widetilde{S} = \big\{\big(F(u,v,\alpha_1,\beta_1,s), G(u,v,\alpha_2,\beta_2,s), H(u,v,\alpha_3,\beta_3,s)\big)\big\}$, then a natural question to ask is whether S is topologically equivalent to $\widetilde{S}$.

Other questions could also be posed regarding the roles of both the scaling factor s and the coordinate systems used for parametrizing the surfaces. In the following examples, we will explore the importance of choosing proper ranges for u and v, as well as specific angular values for the $\alpha_i's$ and β_i's, before determining if two surfaces in (7) and (18) are topologically equivalent.

Example 7 *Consider a cylinder of radius r_1 parametrized by*

$$S=\{(f(u,v),g(u,v),h(u,v))\}=\{(r_1\cos u,r_1\sin u,v)\mid u\in[0,2\pi),v\in[-1,1]\}\,. \qquad (19)$$

Let us remark that the parametrization of the cylinder (19) is performed using cylindrical coordinates (r,u,v), which are related to Cartesian coordinates (x,y,z) by the rule

$$(r,u,v)\quad\longrightarrow\quad (x(r,u,v),y(r,u,v),z(r,u,v))=(r\cos u,r\sin u,v). \qquad (20)$$

The Jacobian matrix corresponding to this change of coordinates (20) is given by

$$\frac{\partial(x,y,z)}{\partial(r,u,v)}=\begin{pmatrix}\cos u & -r\sin u & 0\\ \sin u & r\cos u & 0\\ 0 & 0 & 1\end{pmatrix},$$

whose determinant is

$$\left|\frac{\partial(x,y,z)}{\partial(r,u,v)}\right|=r.$$

Therefore, the Inverse Function Theorem can be applied everywhere, except at $r=0$. That is, any neighborhood of a point (r_0,u_0,v_0), that does not contain the origin $(0,0,0)$, is mapped homeomorphically onto a neighborhood of $(x(r_0,u_0,v_0),y(r_0,u_0,v_0),z(r_0,u_0,v_0))$ by (20) (in fact, it is mapped diffeomorphically).

Now, setting $r_1=1$ in (19), we get the parametrized cylinder

$$S=\{(\cos u,\sin u,v)\mid u\in[0,2\pi],v\in[-1,1]\}\,. \qquad (21)$$

Thus, applying (8) and (9) to (21) with parameter values $\alpha_1=\pi/4$, $\alpha_2=\pi/3$, $\alpha_3=\pi/6$ and $\beta_1=0$, $\beta_2=0$, and $\beta_3=1$, we get two new surfaces

$$S_1=\{(\cos(u+\pi/4),\sin(u+\pi/3),v)\mid u\in[0,2\pi],v\in[-1,1]\} \qquad (22)$$

and

$$S_2=\{(\cos u,\sin u,v+1)\mid u\in[0,2\pi],v\in[-1,1]\}\,. \qquad (23)$$

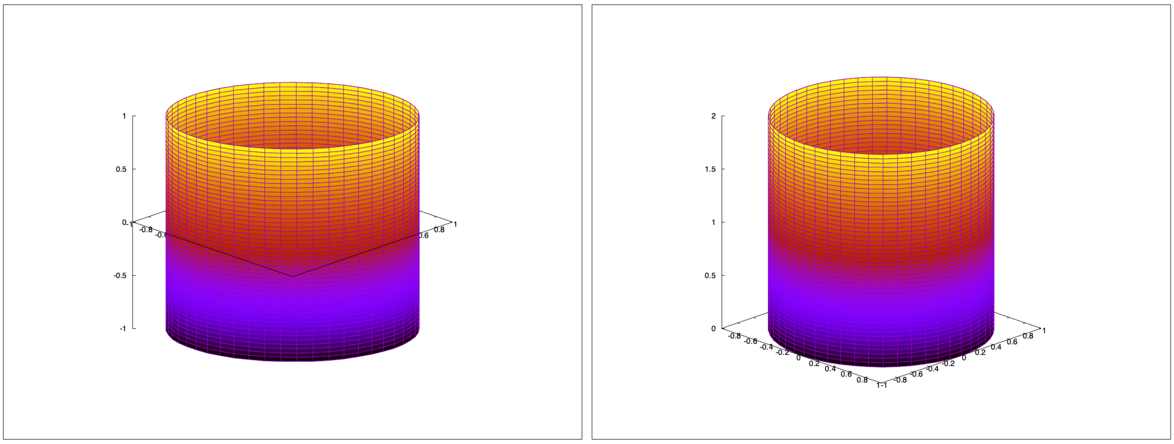
Notice that only two angular values α_1, α_2, and only one height value, β_3, are needed for S_1 and S_2. Figure 10 illustrates both surfaces as parametrized by (22) and (23), respectively.

On the other hand, if we consider transformation (18), then the new surface obtained is parametrized by

$$\widetilde{S}=\left\{s\begin{bmatrix}\cos u\\ \sin u\\ v\end{bmatrix}+(1-s)\begin{bmatrix}\cos(u+\pi/4)\\ \sin(u+\pi/3)\\ v+1\end{bmatrix}\right\}. \qquad (24)$$

Remark that for $s=0$ in (24), we get a new surface $\widetilde{S}$, whereas for $s=1$ we get the original cylinder (21). In Figure 11, we illustrate the transformation of cylinder S, in (21), into the surface $\widetilde{S}$, given by (24), for different values of the parameter s.

We see in this case, that the transformed surface $\widetilde{S}$ is also a cylinder which is moved vertically according to the values of the height parameter v versus that of β_3. The noticeable feature of this process is that the original cylinder S is topologically equivalent to each one of the transformed cylinders for the different values of s.



(a) Surface S_1, as in (8), with $\alpha_1 = \pi/4$, $\alpha_2 = \pi/3$, and $\alpha_3 = \pi/6$.

(b) Surface S_2, as in (9), with $\beta_1 = 0$, $\beta_2 = 0$, and $\beta_3 = 1$.

Figure 10: Two cylinders by perturbing the parametrization, as in (22) and (23).

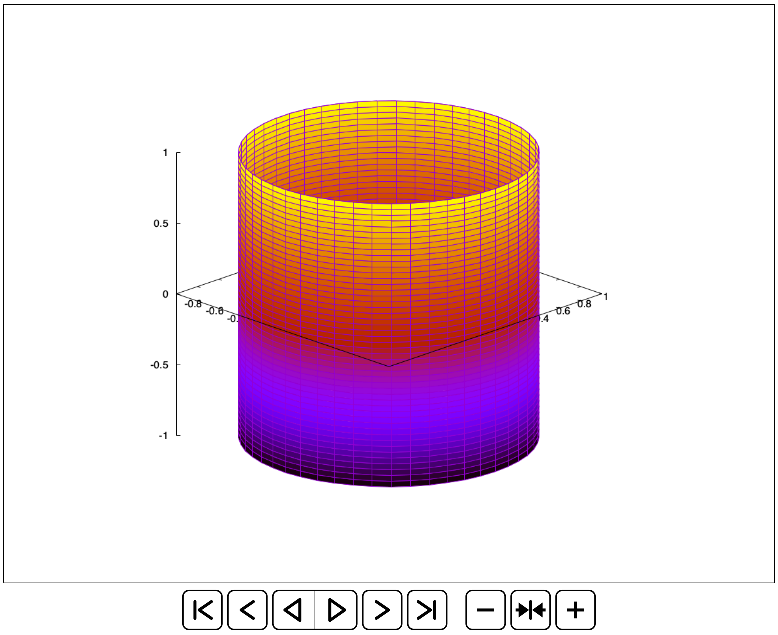


Figure 11: A cylinder is transformed into another cylinder by transformation rule (24).

Example 8 *Let us start with the surface of a parametrized ellipsoid*

$$S = \{(f(u,v), g(u,v), h(u,v))\} = \left\{ \begin{bmatrix} a\sin(v)\cos(u) \\ b\sin(v)\sin(u) \\ c\cos(v) \end{bmatrix} \mid u \in [0, 2\pi], v \in [0, \pi], a, b, c > 0 \right\}. \tag{25}$$

For simplicity, we set $a = b = 1$, and $c = 1.5$ in (25), so we have an ellipsoid parametrized as

$$S = \{(f(u,v), g(u,v), h(u,v))\} = \left\{ \begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ 1.5\cos(v) \end{bmatrix} : 0 \le u \le 2\pi, \ 0 \le v \le \pi \right\}. \tag{26}$$

Notice that the parametrization of the ellipsoid (26) is defined using spherical coordinates. Here, u and v play the role of azimuthal and polar angles, respectively.

Now, according to transformation (18) we have a new surface defined by the set of points of the form

$$\widetilde{S} = \left\{ s \begin{bmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ 1.5\cos(v) \end{bmatrix} + (1-s) \begin{bmatrix} \cos(u+\alpha_1)\sin(v+\beta_1) \\ \sin(u+\alpha_2)\sin(v+\beta_2) \\ 1.5\cos(v+\beta_3) \end{bmatrix} \right\}. \tag{27}$$

Thus, setting particular angular values for the α_i's and the β_i's, say,

$$\begin{aligned} \alpha_1 &= \pi/4, & \beta_1 &= \pi/2, \\ \alpha_2 &= \pi/3, & \beta_2 &= \pi/3, \\ \alpha_3 &= \pi/6, & \beta_3 &= \pi/4, \end{aligned} \tag{28}$$

then, the new surface $\widetilde{S}$ in (27) undergoes a quite interesting behaviour for the different values of the parameter s, as illustrated in the animated graphic of Figure 12.

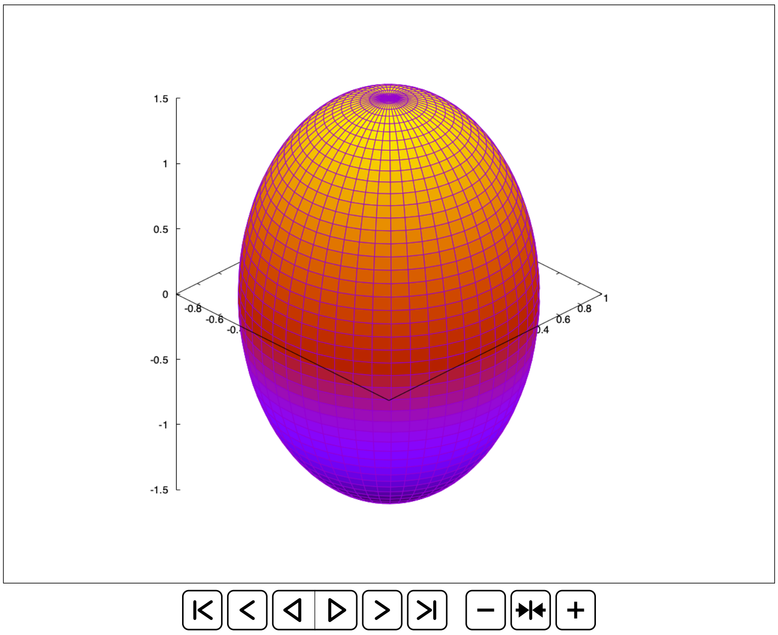


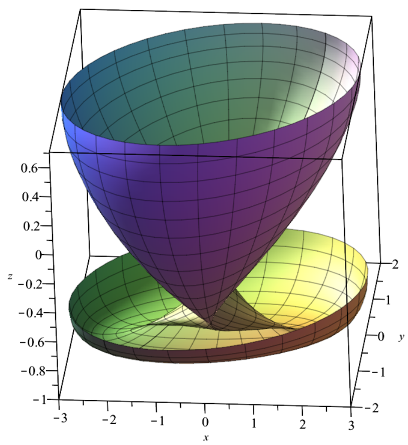
Figure 12: An ellipsoid is transformed according to transformation rule (27).

We can see how remarkable the changes in the transformed surface $\widetilde{S}$ of that such transformations are. We see $\widetilde{S}$ changes the topological properties of the original ellipsoid when $s \neq 1$. In summary, S and $\widetilde{S}$ are not topologically equivalent for $s \neq 1$.

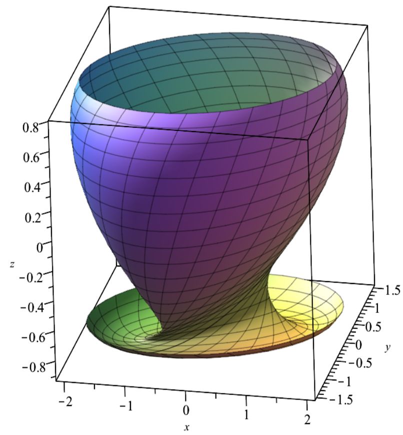
4.2.1 Similar Exploration using Maple™

Considering the transformation in (18) again, we assume the following values: $\alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{3}, \alpha_3 = \frac{\pi}{6}, \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{3}$, and $\beta_3 = \frac{\pi}{4}$. For $u \in [0, 2\pi]$ and $v \in [0, \pi]$, we leave it to the readers to explore and discover for themselves that the surface of points $\begin{bmatrix} F(u,v,\alpha_1,\beta_1,s) \\ G(u,v,\alpha_2,\beta_2,s) \\ H(u,v,\alpha_3,\beta_3,s) \end{bmatrix}$

could not be topologically equivalent to an ellipsoid or a sphere, unless $s = 1$ or $\beta_i = 0$ for all $i = 1, 2, 3$. We depict pictures to illustrate the different figures arising from various values of s in Figures 13a and 13b below. See [S8] or [S9] for further explorations.



(a) The surface when $s = 0$.



(b) Transformation wen $s = 1/3$.

Figure 13: Explorations performed with Maple™

5 Two copies of surfaces when considering $u \in [0, 2\pi]$ and $v \in [0, 2\pi]$

1. If we consider the parameters $u \in [0, 2\pi]$ and $v \in [0, 2\pi]$, with $s = 1$, we can visualize the surface of points

$$\begin{bmatrix} F(u, v, \alpha_1, \beta_1, 1) \\ G(u, v, \alpha_2, \beta_2, 1) \\ H(u, v, \alpha_3, \beta_3, 1) \end{bmatrix} = \begin{bmatrix} f(u, v) \\ g(u, v) \\ h(u, v) \end{bmatrix} = \begin{bmatrix} 5\sin(v)\cos(u); \\ 4\sin(v)\sin(u) \\ 3\cos(v) \end{bmatrix},$$

   to be two copies of the ellipsoid S, which we denote by S^*.

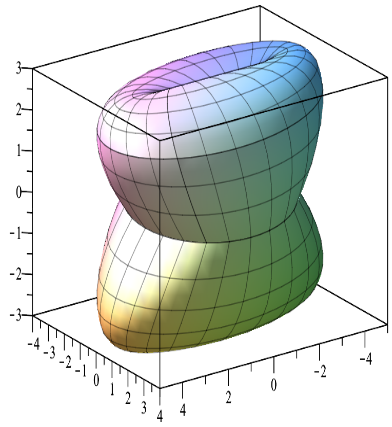
2. However, as we vary the scalar s, we shall investigate the respective surfaces further. For example, when $s = \frac{1}{6}$, the surface

$$S_1 = \left\{ \begin{bmatrix} F(u, v, \alpha_1, \beta_1, \frac{1}{6}) \\ G(u, v, \alpha_2, \beta_2, \frac{1}{6}) \\ H(u, v, \alpha_3, \beta_3, \frac{1}{6}) \end{bmatrix} \;\middle|\; u, v \in [0, 2\pi] \right\}, \tag{29}$$

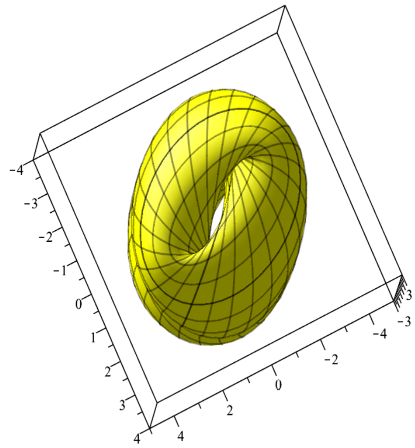
   shown in Figure 14a is a surface for which we are interested in knowing if it is topologically equivalent to S^*. Thus, we investigate the surface more closely as follows:

   (a) For $s = \frac{1}{6}$, we think of S_1 as the union of two copies of

$$S_1^t = \left\{ \begin{bmatrix} F(u, v, \alpha_1, \beta_1, \frac{1}{6}) \\ G(u, v, \alpha_1, \beta_1, \frac{1}{6}) \\ H(u, v, \alpha_1, \beta_1, \frac{1}{6}) \end{bmatrix} \;\middle|\; u \in [0, 2\pi],\ v \in [0, \pi] \right\} \tag{30}$$



(a) The closed surface S_1.



(b) Not topologically equivalent to S^* for $s = 1/3$.

Figure 14: Some more explorations for an ellipsoid.

and

$$S_1^b = \left\{ \begin{bmatrix} F(u, v, \alpha_1, \beta_1, \frac{1}{6}) \\ G(u, v, \alpha_1, \beta_1, \frac{1}{6}) \\ H(u, v, \alpha_1, \beta_1, \frac{1}{6}) \end{bmatrix} \middle| \; u \in [0, 2\pi],\; v \in [\pi, 2\pi] \right\}, \tag{31}$$

where

$$F(u, v, \alpha_1, \beta_1, \frac{1}{6}) = \frac{5\cos u \sin v + 25\cos(u + \frac{\pi}{4})\cos v}{6}, \tag{32}$$

$$G(u, v, \alpha_1, \beta_1, \frac{1}{6}) = \frac{2\sin u \sin v + 10\sin(u + \frac{\pi}{3})\sin(v + \frac{\pi}{3})}{3}, \tag{33}$$

$$H(u, v, \alpha_1, \beta_1, \frac{1}{6}) = \frac{\cos v + 5\cos(v + \frac{\pi}{4})}{2}. \tag{34}$$

Thus, we have

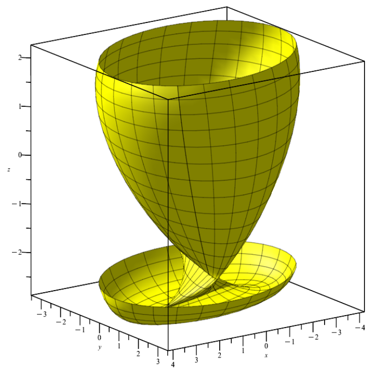
$$S_1 = S_1^t \cup S_1^b, \tag{35}$$

and we can think of S_1 (see Figure 15c) as the union of two tea-cups (S_1^t is the yellow picture in Figure 15a, and S_1^b is the blue picture in Figure 15b).

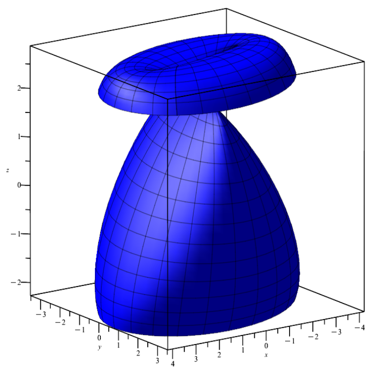
(b) We see, since S_1^t and S_1^b are not closed surfaces, neither is topologically equivalent to a sphere or an ellipsoid. However, S_1 is a closed surface. It is still not known if S_1 (shown in Figure 14a) can be reparameterized so that it is topologically equivalent to a sphere or an ellipsoid. We encourage interested readers to prove or disprove. It is also beyond the scope of this paper to discuss the orientability for S_1^t or S_1^b. Readers can explore further in [S8]. In particular, the animations of the space curves for the longitudes (when $u \in [0, 2\pi]$), and latitudes for S_1 (when $v \in [0, 2\pi]$), respectively, provide us interesting graphical representations on how S_1 is generated:

i. `animate(plot3d, [[F1(u, v, 1/6), G1(u, v, 1/6), H1(u, v, 1/6)], u = 0 .. 2*Pi, v = 0 ..  t], t = 0 ..  2*Pi, thickness = 3, color = yellow);`

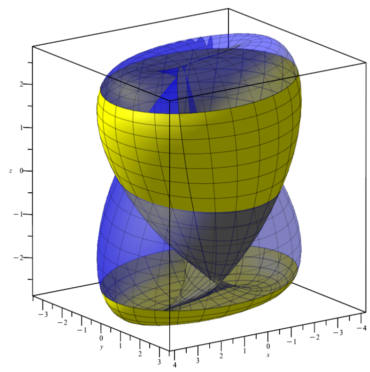
ii. `animate(plot3d, [[F1(u, v, 1/6), G1(u, v, 1/6), H1(u, v, 1/6)], u = 0 .. t, v = 0 ..  2*Pi], t = 0 ..  2*Pi, thickness = 3, color = yellow);`



(a) Graph of S_1^t.



(b) Graph of S_1^b.



(c) Graph of $S_1^t \cup S_1^b$.

Figure 15: Two copies of a surface for $s = 1/6$.

iii. See [S10] and [S11], respectively, for the animations.

3. When $s = \frac{1}{3}$, the surface

$$\left\{ \begin{bmatrix} F(u, v, \alpha_1, \beta_1, \frac{1}{3}) \\ G(u, v, \alpha_2, \beta_2, \frac{1}{3}) \\ H(u, v, \alpha_3, \beta_3, \frac{1}{3}) \end{bmatrix} \middle| \, u, v \in [0, 2\pi] \right\}$$

is a surface (shown in Figure 14b) that is clearly not topologically equivalent to S^*. Therefore, for a given set of values of α_i and β_i, one may ask how to find the maximum value of $s \in (0, 1)$ such that the surface

$$\left\{ \begin{bmatrix} F(u, v, \alpha_1, \beta_1, s) \\ G(u, v, \alpha_2, \beta_2, s) \\ H(u, v, \alpha_3, \beta_3, s) \end{bmatrix} \middle| \, u, v \in [0, 2\pi] \right\}$$

is still topologically equivalent to the original ellipsoid. See [S7] or [S8] for explorations.

6 Future works

There are several instances one can investigate further, which we describe below:

1. Suppose we are given a surface Σ and a point $C \in \Sigma$, with D its the antipodal point when connected by a line l passing through point C and a fixed point A. In this situation, we shall explore the following scenario: for a given Σ we select three vectors AB, AC and AD, defined by the three points B, C and D, respectively, which lie on surface Σ. We will investigate the newly generated affine surface

$$\Sigma' = r \cdot AB + s \cdot AC + t \cdot AD, \tag{36}$$

where $r, s, t \in \mathbb{R}$, satisfying $r + s + t = 1$.

2. One can investigate a new surface that is generated by three vectors that satisfies an affine combination. We start with a surface S of points $\begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix}$, then by picking particular angles α and β respectively, we form the new generated surface via the following transformation:

$$\begin{bmatrix} F(u,v,\alpha,\beta) \\ G(u,v,\alpha,\beta) \\ H(u,v,\alpha,\beta) \end{bmatrix} = r \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} + s \begin{bmatrix} f(u+\alpha,v) \\ g(u+\alpha,v) \\ h(u+\alpha,v) \end{bmatrix} + t \begin{bmatrix} f(u,v+\beta) \\ g(u,v+\beta) \\ h(u,v+\beta) \end{bmatrix}, \qquad (37)$$

where $r, s, t \in \mathbb{R}$ with the **affine combination condition** of

$$r + s + t = 1. \qquad (38)$$

We believe that such transformation may find uses in computer graphics and other applications because of the scaling factors of r, s, t and rotation factors of angles α and β.

We can also view the newly generated surface as a linear combination of the vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$, where

$$\begin{aligned} \overrightarrow{OP} &= \big[f(u,v), g(u,v), h(u,v)\big] \\ \overrightarrow{OQ} &= \big[f(u+\alpha,v), g(u+\alpha,v), g(u+\alpha,v)\big] \\ \overrightarrow{OR} &= \big[f(u,v+\beta), g(u,v+\beta), h(u,v+\beta)\big]. \end{aligned}$$

3. We may investigate topological surfaces, such as the torus, under the transformation defined by 37. For example, we consider a torus parametrized by

$$\begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} = \begin{bmatrix} (a + b\cos u)\cos v \\ (a + b\cos u)\sin v \\ c\sin u \end{bmatrix}$$

for $u \in [0, 2\pi)$, $v \in [0, 2\pi)$, and consider the surface generated by the points

$$\begin{bmatrix} F(u,v,\alpha,\beta) \\ G(u,v,\alpha,\beta) \\ H(u,v,\alpha,\beta) \end{bmatrix} = r \begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix} + s \begin{bmatrix} f(u+\alpha,v) \\ g(u+\alpha,v) \\ h(u+\alpha,v) \end{bmatrix} + t \begin{bmatrix} f(u,v+\beta) \\ g(u,v+\beta) \\ h(u,v+\beta) \end{bmatrix},$$

where $\alpha \in (0, 2\pi)$, $\beta \in \left(0, \frac{\pi}{2}\right)$, $r, s, t \in \mathbb{R}$, with $r + s + t = 1$.

7 Conclusions

It is exciting to see that a series of papers, [9], [10], [11], [12], and [13], which was originated from a simple college entrance exam from China, has utilized ideas from widely-ranging fields of mathematics, such as algebraic geometry, projective geometry, differential geometry and so on. Many will agree that one objective of exploring mathematics with technological tools is to inspire undergraduate or graduate students to expand their content knowledge to broaden their mathematical horizons.

It is clear that technological tools provide us with many crucial and much-needed intuitions before seeking for more rigorous analytical solutions. In the meantime, we use a CAS, for verifying that our analytical solutions are consistent with our initial intuitions. The complexity level of the problems we posed vary from simple to difficult. Many of our solutions are accessible to students from high school, and thus, serve as excellent exemplars used in teachers' professional development.

Evolving technological tools not only make mathematics fun and accessible on one hand, but also allow the exploration of more challenging and theoretical mathematics on the other. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to carry out investigations in problems of varying levels of difficulty. Inculcating a greater interest in mathematics for students, and in particular providing them with the technological tools to explore challenging and intricate problems beyond the reach of pencil-and-paper is an important step for cultivating creativity and innovation.

8 Acknowledgements

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9 Supplementary Electronic Materials

[S1] GeoGebra worksheet, SuplMat2.ggb.

[S2] GeoGebra worksheet, SuplMat5.ggb.

[S3] GeoGebra worksheet, SuplMat6.ggb

[S4] GeoGebra worksheet, SuplMat1.ggb.

[S5] Animation for Example 5, Example5.gif.

[S6] Maple file for Example 5, Example5-cardioid.mw.

[S7] Animation for Section 5, Section5_part1.gif, when $u, v \in [0, 2\pi]$.

[S8] Maple file for Section 5, Section5.mw.

[S9] Animation for Section 5, Section5_part2.gif, when $v \in [0, \pi]$.

[S10] Longitude animation for S_1, S1_longitude.gif.

[S11] Latitude animation for S_1, S1_latitude.gif.

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[2] Maple, A product of Maplesoft, see http://Maplesoft.com/.

[3] Maxima.sourceforge.net. Maxima, a Computer Algebra System. Version 5.46.0 (2022) https://maxima.sourceforge.io

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[6] Mathworld: https://mathworld.wolfram.com/QuadraticSurface.html

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[13] Yang, W.-C. & Morante, A., *Locus of Antipodal Projection When Fixed Point is Outside a Curve or Surface*, the Electronic Proceedings of the 26th Asian Technology Conference in Mathematics. Published by Mathematics and Technology, LLC, ISSN 1940-4204 (online version).

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